

Cycle packings of the complete multigraph

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Abstract

Bryant, Horsley, Maenhaut and Smith recently gave necessary and sufficient conditions for when the complete multigraph can be decomposed into cycles of specified lengths m_1, m_2, \dots, m_τ . In this paper we characterise exactly when there exists a packing of the complete multigraph with cycles of specified lengths m_1, m_2, \dots, m_τ . While cycle decompositions can give rise to packings by removing cycles from the decomposition, in general it is not known when there exists a packing of the complete multigraph with cycles of various specified lengths.

1 Introduction

A *decomposition* of a multigraph G is a collection \mathcal{D} of submultigraphs of G such that each edge of G is in exactly one of the multigraphs in \mathcal{D} . A *packing* of a multigraph G is a collection \mathcal{P} of submultigraphs of G such that each edge of G is in at most one of the multigraphs in \mathcal{P} . The *leave* of a packing \mathcal{P} is the multigraph obtained by removing the edges in multigraphs in \mathcal{P} from G . A *cycle packing* of a multigraph G is a packing \mathcal{P} of G such that each submultigraph in \mathcal{P} is a cycle. For positive integers λ and v , λK_v denotes the complete multigraph with λ parallel edges between each pair of v distinct vertices. Here we give a complete characterisation of when there exists a packing of λK_v with cycles of specified lengths m_1, \dots, m_τ . Note that for $\lambda \geq 2$, λK_v contains 2-cycles (pairs of parallel edges).

Theorem 1. *Let m_1, \dots, m_τ be a nondecreasing list of integers and let λ and v be positive integers. Then there exists a packing of λK_v with cycles of lengths m_1, \dots, m_τ if and only if*

- (i) $2 \leq m_1, \dots, m_\tau \leq v$;
- (ii) $m_1 + \dots + m_\tau = \lambda \binom{v}{2} - \delta$, where δ is a nonnegative integer such that $\delta \neq 1$ when $\lambda(v-1)$ is even, $\delta \neq 2$ when $\lambda = 1$, and $\delta \geq \frac{v}{2}$ when $\lambda(v-1)$ is odd;
- (iii) $\sum_{m_i=2} m_i \leq \begin{cases} (\lambda-1)\binom{v}{2} - 2 & \text{if } \lambda \text{ and } v \text{ are odd and } \delta = 2, \\ (\lambda-1)\binom{v}{2} & \text{if } \lambda \text{ is odd; and} \end{cases}$
- (iv) $m_\tau \leq \begin{cases} \frac{\lambda}{2}\binom{v}{2} - \tau + 2 & \text{if } \lambda \text{ is even and } \delta = 0, \\ \frac{\lambda}{2}\binom{v}{2} - \tau + 1 & \text{if } \lambda \text{ is even and } 2 \leq \delta < m_\tau. \end{cases}$

Bryant, Horsley, Maenhaut and Smith [5] recently characterised exactly when there exists a decomposition of the complete multigraph λK_v into cycles of specified lengths m_1, \dots, m_τ (see also [4, 12]). Since a decomposition of a multigraph is a packing whose leave contains no edges, many instances of the cycle packing problem can be solved by removing cycles from a cycle decomposition λK_v . However there are cases which cannot be solved in this manner. These cases occur when $\lambda(v-1)$ is odd and there are $\frac{v}{2} + 1$ or $\frac{v}{2} + 2$ edges in the leave of the required packing.

In the case of the complete graph K_v (with $\lambda = 1$), it had previously been found exactly when there exist decompositions into cycles of specified lengths [6]. Furthermore, Horsley [10] found conditions for the existence of packings of the complete graph with uniform length cycles. These results built on earlier results for cycle decompositions and packings of the complete graph [1, 2, 9, 11] (see [7] for a survey). However, even in the $\lambda = 1$ case, a complete characterisation of when there exists a packing of K_v with cycles of lengths m_1, \dots, m_τ had not previously been obtained.

We will show that the necessity of conditions (i)–(iv) in Theorem 1 follows from known results for cycle decompositions of λK_v . The sufficiency of these conditions is proved by first decomposing λK_v into cycles (and a 1-factor if $\lambda(v-1)$ is even) and then removing cycles and modifying the resulting packing to obtain the one that we require. The existence of these cycle decompositions of λK_v was obtained by Bryant et al [5] and the exact result is stated as Theorem 5 in Section 3. Section 2 contains the results required for modifying cycle decompositions.

The following definitions and notation will be used throughout this paper. An (m_1, \dots, m_τ) -decomposition of λK_v is a decomposition of λK_v into τ cycles of lengths m_1, \dots, m_τ . Similarly, an (m_1, \dots, m_τ) -packing of λK_v is a packing of λK_v with τ cycles of lengths m_1, \dots, m_τ . For vertices x and y in a multigraph G , the *multiplicity* of xy is the number of edges in G which have x and y as their endpoints, denoted $\mu_G(xy)$. If $\mu_G(xy) \leq 1$ for all pairs of vertices in $V(G)$ then we say that G is a *simple* graph. A multigraph is said to be *even* if every vertex has even degree and is said to be *odd* if every vertex has odd degree.

Given a permutation π of a set V , a subset S of V and a multigraph G with $V(G) \subseteq V$, $\pi(S)$ is defined to be the set $\{\pi(x) : x \in S\}$ and $\pi(G)$ is defined to be the multigraph with vertex set $\pi(V(G))$ and edge set $\{\pi(x)\pi(y) : xy \in E(G)\}$. The m -cycle with vertices x_0, x_1, \dots, x_{m-1} and edges $x_i x_{i+1}$ for $i \in \{0, \dots, m-1\}$ (with subscripts modulo m) is denoted by $(x_0, x_1, \dots, x_{m-1})$ and the n -path with vertices y_0, y_1, \dots, y_n and edges $y_j y_{j+1}$ for $j \in \{0, 1, \dots, n-1\}$ is denoted by $[y_0, y_1, \dots, y_n]$.

A *chord* of a cycle is an edge which is incident with two vertices of the cycle but is not in the cycle. Note that a chord may be an edge parallel to an edge of the cycle. For integers $p \geq 2$ and $q \geq 1$, a (p, q) -*lasso* is the union of a p -cycle and a q -path such that the cycle and the path share exactly one vertex and that vertex is an end-vertex of the path. A (p, q) -lasso with cycle (x_1, x_2, \dots, x_p) and path $[x_p, y_1, y_2, \dots, y_q]$ is denoted by $(x_1, x_2, \dots, x_p)[x_p, y_1, y_2, \dots, y_q]$. The *order* of a (p, q) -lasso is $p + q$.

2 Modifying cycle packings of λK_v

The aim of this section is to prove Lemmas 3 and 4. These results are useful tools for modifying cycle packings of the complete multigraph. The simple graph versions of Lemmas 3 and 4 are

due to Bryant and Horsley [8] and have been applied to prove the maximum packing result of the simple complete graph with uniform length cycles [10].

We require the following cycle switching lemma for cycle packings of multigraphs. Lemma 2 is similar to [4, Lemma 2.1] and is also closely related to the cycle switching method which has been applied to simple graphs (see for example [3]).

Lemma 2. *Let v and λ be positive integers, let M be a list of integers, let \mathcal{P} be an (M) -packing of λK_v , let L be the leave of \mathcal{P} , let α and β be distinct vertices of L , and let π be the transposition $(\alpha\beta)$. Let E be a subset of $E(L)$ such that, for each vertex $x \in V(L) \setminus \{\alpha, \beta\}$, E contains precisely $\max(0, \mu_L(x\alpha) - \mu_L(x\beta))$ edges with endpoints x and α , and precisely $\max(0, \mu_L(x\beta) - \mu_L(x\alpha))$ edges with endpoints x and β (so E may contain multiple edges with the same endpoints), and E contains no other edges. Then there exists a partition of E into pairs such that for each pair $\{x_1y_1, x_2y_2\}$ of the partition, there exists an (M) -packing \mathcal{P}' of λK_v with leave $L' = (L - \{x_1y_1, x_2y_2\}) + \{\pi(x_1)\pi(y_1), \pi(x_2)\pi(y_2)\}$.*

Furthermore, if $\mathcal{P} = \{C_1, \dots, C_t\}$, then $\mathcal{P}' = \{C'_1, \dots, C'_t\}$ where for $i \in \{1, \dots, t\}$, C'_i is a cycle of the same length as C_i such that for $i \in \{1, \dots, t\}$

- *If neither α nor β is in $V(C_i)$, then $C'_i = C_i$;*
- *If exactly one of α and β is in $V(C_i)$, then $C'_i = C_i$ or $C'_i = \pi(C_i)$; and*
- *If both α and β are in $V(C_i)$, then $C'_i = Q_i \cup Q_i^*$ where $Q_i = P_i$ or $\pi(P_i)$, $Q_i^* = P_i^*$ or $\pi(P_i^*)$, and P_i and P_i^* are the two paths from α to β in C_i .*

Proof. When $\lambda(v-1)$ is even, Lemma 2 is identical to [4, Lemma 2.1] so this case has already been proved. Lemma 2 differs from [4, Lemma 2.1] in that here \mathcal{P} is a cycle packing of λK_v regardless of the parity of $\lambda(v-1)$, whereas when $\lambda(v-1)$ is odd [4, Lemma 2.1] concerns a cycle packing of $\lambda K_v - I$, where I is a 1-factor of λK_v . However, in this case the proof of Lemma 2 follows from very similar arguments to those used in the corresponding case of the proof in [4]. \square

In applying Lemma 2 we say that we are performing the (α, β) -switch with origin x and terminus y (where $\{x_1, y_1, x_2, y_2\} \subseteq \{\alpha, \beta, x, y\}$). Note that x_1y_1 and x_2y_2 may be parallel edges, in which case $x = y$.

Lemma 3. *Let v , s and λ be positive integers such that $s \geq 3$, and let M be a list of integers. Suppose there exists an (M) -packing \mathcal{P} of λK_v whose leave contains a lasso of order at least $s+2$ and suppose that if s is even then the cycle of the lasso has even length. Then there exists an (M, s) -packing of λK_v .*

Proof. Let L be the leave of \mathcal{P} . Suppose that L contains a (p, q) -lasso $(x_1, x_2, \dots, x_p)[x_p, y_1, y_2, \dots, y_q]$ such that $p+q \geq s+2$ and p is even if s is even. If L contains an s -cycle then we add it to the packing to complete the proof, so assume L does not contain an s -cycle and hence $p \neq s$.

Case 1. Suppose $2 \leq p < s$ and either $p = 2$ or $p \equiv s \pmod{2}$. We can assume that $p+q = s+2$ since L contains a $(p, s+2-p)$ -lasso.

Let L' be the leave of the packing \mathcal{P}' obtained from \mathcal{P} by applying the (x_1, y_{q-1}) -switch with origin x_2 (note that $\mu_L(x_2y_{q-1}) = 0$ for otherwise L contains an s -cycle). If the terminus of the switch is not y_{q-2} then L' contains an s -cycle which completes the proof (recall that

$s = p + q - 2$). Otherwise, the terminus of the switch is y_{q-2} and L' contains a (q, p) -lasso $(x'_1, x'_2, \dots, x'_q)[x'_q, y'_1, y'_2, \dots, y'_p]$. If $p = 2$ then L' contains an s -cycle which completes the proof, so assume L' contains no s -cycle and $p \geq 3$.

Let L'' be the leave of the packing \mathcal{P}'' obtained from \mathcal{P}' by applying the (x'_2, y'_p) -switch with origin x'_3 (note that $\mu_{L'}(x'_3 y'_p) = 0$ for otherwise L' contains an s -cycle). If the terminus of this switch is not y'_{p-1} then L'' contains an s -cycle which completes the proof (recall that $s = p + q - 2$). Otherwise, the terminus of the switch is y'_{p-1} and L'' contains a $(p+2, q-2)$ -lasso, so since $p < s$ and $p \equiv s \pmod{2}$, the result follows by repeating the procedure described in this case.

Case 2. Suppose $3 \leq p < s$ and $p \not\equiv s \pmod{2}$. As above, assume $p + q = s + 2$. Then s is odd, $p \geq 4$ is even and q is odd by our hypotheses.

Let L' be the leave of the packing \mathcal{P}' obtained from \mathcal{P} by applying the (x_2, y_q) -switch with origin x_3 (note that $\mu_L(x_3 y_q) = 0$ for otherwise L contains an s -cycle). If the terminus of the switch is not y_{q-1} then L' contains an s -cycle which completes the proof. Otherwise, the terminus of the switch is y_{q-1} and L' contains a $(q+2, p-2)$ -lasso. Note that $q+2 \leq s$ (because $p + q = s + 2$ and $p \geq 4$) and $q+2 \equiv s \pmod{2}$. If $q+2 = s$ then this completes the proof, otherwise we can proceed as in Case 1.

Case 3. Suppose $3 \leq s < p$. Let L' be the leave of the packing \mathcal{P}' obtained from \mathcal{P} by applying the (x_{p-s+1}, y_1) -switch with origin x_{p-s+2} (note that $\mu_L(x_{p-s+2} y_1) = 0$ for otherwise L contains an s -cycle). If the terminus of the switch is not x_p then L' contains an s -cycle which completes the proof. Otherwise, L' contains a $(p-s+2, q+s-2)$ -lasso. By repeating this process we obtain an (M) -packing of λK_v whose leave contains a $(p', p+q-p')$ -lasso such that $2 \leq p' \leq s$ and $p' \equiv p \pmod{s+2}$. If $p' = s$ then this completes the proof, otherwise we can proceed as in Case 1 or Case 2. \square

Lemma 4. Let v , s and λ be positive integers with $s \geq 3$, and let M be a list of integers. Suppose there exists an (M) -packing of λK_v whose leave L has a component H containing an $(s+1)$ -cycle with a chord. Then there exists an (M) -packing of λK_v with a leave L' such that $E(L') = (E(L) \setminus E(H)) \cup E(H')$, where H' is a graph with $V(H') = V(H)$ and $|E(H')| = |E(H)|$ which contains an $(s, 1)$ -lasso. Furthermore, $\deg_{H'}(x) \geq \deg_H(x)$ for each vertex x in the s -cycle of this lasso.

Proof. Let (x_1, \dots, x_{s+1}) be an $(s+1)$ -cycle in H with chord $x_1 x_e$ for some $e \in \{2, 3, \dots, s-1\}$ (note that L is not necessarily a simple graph). If H contains an $(s, 1)$ -lasso then we are finished immediately, so suppose otherwise. If $e = 2$, then perform the (x_3, x_2) -switch with origin x_4 (note that $\mu_L(x_2 x_4) = 0$ because H contains no $(s, 1)$ -lasso). The leave of the resulting packing contains the $(s, 1)$ -lasso $(x_4, \dots, x_{s+1}, x_1, x_2)[x_2, x_3]$, and $\deg_{H'}(x_i) \geq \deg_H(x_i)$ for $i \in \{1, \dots, s+1\} \setminus \{3\}$. If $e = 3$, then H contains an $(s, 1)$ -lasso which completes the proof.

So suppose $e \geq 4$ and let \mathcal{P}^* be the packing with leave L^* obtained from \mathcal{P} by applying the (x_{e-1}, x_e) -switch with origin x_{e-2} (note that $\mu_L(x_{e-2} x_e) = 0$ for otherwise L contains an $(s, 1)$ -lasso). If the terminus of the switch is not x_{e+1} then $E(L^*) = (E(L) \setminus E(H)) \cup E(H^*)$, where H^* is a graph with $V(H^*) = V(H)$ and $|E(H^*)| = |E(H)|$ which contains the $(s, 1)$ -lasso $(x_{e+1}, \dots, x_{s+1}, x_1, \dots, x_{e-2}, x_e)[x_e, x_{e-1}]$. Also note that $\deg_{H^*}(x_e) \geq \deg_H(x_e)$ and $\deg_{H^*}(x_i) = \deg_H(x_i)$ for $i \in \{1, \dots, s+1\} \setminus \{e, e-1\}$. Otherwise, the terminus of the switch is x_{e+1} and $E(L^*) = (E(L) \setminus E(H)) \cup E(H^*)$, where H^* is a graph with $V(H^*) = V(H)$ and $|E(H^*)| = |E(H)|$ which contains an $(s+1)$ -cycle $(x_1^*, \dots, x_{s+1}^*)$ with chord $x_1^* x_{e-1}^*$.

Furthermore, the degree of each vertex in this $(s + 1)$ -cycle remains unchanged in H' . The result follows by repeating this process. \square

3 Main result

This section contains the proof of Theorem 1. We first use Theorem 5 (stated below) to prove Lemma 6 which shows the necessity of the conditions in Theorem 1. The sufficiency of these conditions is then established for λ odd and λ even in Lemmas 7 and 8 respectively. Lemmas 7 and 8 rely on using Lemmas 3 and 4 to modify cycle packings of λK_v obtained via Theorem 5.

Theorem 5 ([5]). *Let m_1, \dots, m_τ be a nondecreasing list of integers and let λ and v be positive integers. There is an (m_1, \dots, m_τ) -decomposition of λK_v if and only if*

- $\lambda(v - 1)$ is even;
- $2 \leq m_1, \dots, m_\tau \leq v$;
- $m_1 + \dots + m_\tau = \lambda \binom{v}{2}$;
- $m_\tau + \tau - 2 \leq \frac{\lambda}{2} \binom{v}{2}$ when λ is even; and
- $\sum_{m_i=2} m_i \leq (\lambda - 1) \binom{v}{2}$ when λ is odd.

There is an (m_1, \dots, m_τ) -decomposition of $\lambda K_v - I$, where I is a 1-factor in λK_v , if and only if

- $\lambda(v - 1)$ is odd;
- $2 \leq m_1, \dots, m_\tau \leq v$;
- $m_1 + \dots + m_\tau = \lambda \binom{v}{2} - \frac{v}{2}$; and
- $\sum_{m_i=2} m_i \leq (\lambda - 1) \binom{v}{2}$.

The necessity of conditions Theorem 1(i)–(iv) follows from Theorem 5 as we now show.

Lemma 6. *Let m_1, \dots, m_τ be a nondecreasing list of integers and let λ and v be positive integers. If there exists an (m_1, \dots, m_τ) -packing of λK_v then*

- (i) $2 \leq m_1, \dots, m_\tau \leq v$;
- (ii) $m_1 + \dots + m_\tau = \lambda \binom{v}{2} - \delta$, where δ is a nonnegative integer such that $\delta \neq 1$ when $\lambda(v - 1)$ is even, $\delta \neq 2$ when $\lambda = 1$, and $\delta \geq \frac{v}{2}$ when $\lambda(v - 1)$ is odd;
- (iii) $\sum_{m_i=2} m_i \leq \begin{cases} (\lambda - 1) \binom{v}{2} - 2 & \text{if } \lambda \text{ and } v \text{ are odd and } \delta = 2, \\ (\lambda - 1) \binom{v}{2} & \text{if } \lambda \text{ is odd; and} \end{cases}$
- (iv) $m_\tau \leq \begin{cases} \frac{\lambda}{2} \binom{v}{2} - \tau + 2 & \text{if } \lambda \text{ is even and } \delta = 0, \\ \frac{\lambda}{2} \binom{v}{2} - \tau + 1 & \text{if } \lambda \text{ is even and } 2 \leq \delta < m_\tau. \end{cases}$

Proof. Suppose there exists an (m_1, \dots, m_τ) -packing \mathcal{P} of λK_v with leave L . Condition (i) is obvious. The degree of each vertex in λK_v is $\lambda(v-1)$, so if $\lambda(v-1)$ is even then L is an even multigraph and if $\lambda(v-1)$ is odd then L is an odd multigraph. Hence (ii) follows because an even graph cannot have a single edge, an even simple graph cannot have two edges, and an odd graph on v vertices has at least $\frac{v}{2}$ edges. To see that condition (iii) holds, note that there are at most $\lfloor \frac{\lambda}{2} \rfloor \binom{v}{2}$ edge-disjoint 2-cycles in λK_v . Furthermore, note that if λ and v are both odd and $\delta = 2$ then L is a 2-cycle (because L is an even multigraph and has two edges). If λ is even and $\delta = 0$ then (iv) follows directly from Theorem 5, so suppose λ is even and $2 \leq \delta < m_\tau$. Then L contains at least one cycle so there exists an (m_1, \dots, m_τ, M) -decomposition of λK_v for some list M containing at least one entry. So (iv) follows from Theorem 5. \square

It remains to prove the sufficiency of Theorem 1(i)–(iv) for the existence of cycle packings of λK_v .

Lemma 7. *Let m_1, \dots, m_τ be a nondecreasing list of integers and let λ and v be positive integers with λ odd. Then there exists an (m_1, \dots, m_τ) -packing of λK_v if and only if*

$$(i) \quad 2 \leq m_1, \dots, m_\tau \leq v;$$

$$(ii) \quad m_1 + \dots + m_\tau = \lambda \binom{v}{2} - \delta, \text{ where } \delta \text{ is a nonnegative integer such that } \delta \neq 1, (\lambda, \delta) \neq (1, 2), \text{ and if } v \text{ is even then } \delta \geq \frac{v}{2}; \text{ and}$$

$$(iii) \quad \sum_{m_i=2} m_i \leq \begin{cases} (\lambda-1) \binom{v}{2} - 2 & \text{if } v \text{ is odd and } \delta = 2, \\ (\lambda-1) \binom{v}{2} & \text{otherwise.} \end{cases}$$

Proof. If there exists an (m_1, \dots, m_τ) -packing \mathcal{P} of λK_v , then conditions (i)–(iii) hold by Lemma 6. So it remains to show that if λ , v and m_1, \dots, m_τ satisfy (i)–(iii), then there is an (m_1, \dots, m_τ) -packing of λK_v .

Let $\varepsilon = \delta$ if v is odd, and $\varepsilon = \delta - \frac{v}{2}$ if v is even. If $\varepsilon = 0$ then the result follows by Theorem 5. If $v = 2$, then ε is even by (i) and (ii) and there exists a 2-cycle decomposition of $\lambda K_2 - I$, where I is a 1-factor of λK_2 , so the result follows. So suppose $\varepsilon \geq 1$ and $v \geq 3$, and note that if v is odd then $\varepsilon \neq 1$ and $(\lambda, \varepsilon) \neq (1, 2)$.

Case 1. Suppose v is odd or $\varepsilon \geq 3$. Note that if v is odd and $\varepsilon = 2$ then $2 + \sum_{m_i=2} m_i \leq (\lambda-1) \binom{v}{2}$ by (iii).

We show that there exists a list N such that $2 \leq n \leq v$ for all $n \in N$, $\sum N = \varepsilon$ and $\sum_{n \in N, n=2} n + \sum_{m_j=2} m_j \leq (\lambda-1) \binom{v}{2}$. If this list exists, then by Theorem 5 there exists an (m_1, \dots, m_τ, N) -decomposition \mathcal{D} of λK_v (if v is odd) or $\lambda K_v - I$ (if v is even), where I is a 1-factor of λK_v . We obtain the required packing by removing cycles of lengths N from \mathcal{D} .

We first consider $v = 3$. If $v = 3$ and ε is even, then $m_i = 3$ for some $i \in \{1, \dots, \tau\}$ by (i) and (ii). Then $\varepsilon + \sum_{m_i=2} m_i \leq (\lambda-1) \binom{v}{2}$ by (ii) and we take $N = (2^{\varepsilon/2})$. If $v = 3$ and ε is odd then $\varepsilon - 3 + \sum_{m_i=2} m_i \leq (\lambda-1) \binom{v}{2}$ by (ii) and we take $N = (2^{(\varepsilon-3)/2}, 3)$. In each of these cases we can see that there exists an (m_1, \dots, m_τ, N) -decomposition of λK_v since the hypotheses of Theorem 5 are satisfied by (i)–(iii).

Now assume $v \geq 4$ and let q and r be nonnegative integers such that $\varepsilon = vq + r$ and $0 \leq r < v$. If $q = 0$ or $r \notin \{1, 2\}$ then we take $N = (r, v^q)$. If $q \geq 1$ and $r \in \{1, 2\}$ then $N = (3, v-3+r, v^{q-1})$ (note that either $v-3+r \geq 3$, or $v = 4$ and $r = 1$). If $\varepsilon = 2$ or $(v, r) = (4, 1)$, then N contains exactly one entry equal to 2 and otherwise $n \geq 3$ for all $n \in N$. By (iii) and the hypotheses of this case, if $\varepsilon = 2$ then $2 + \sum_{m_i=2} m_i \leq (\lambda-1) \binom{v}{2}$.

Further, if $v = 4$ and $\varepsilon = 4q + 1$ for some $q \geq 1$ then (i) and (ii) imply that $m_i = 3$ for some $i \in \{1, \dots, \tau\}$ so again $2 + \sum_{m_i=2} m_i \leq (\lambda - 1)\binom{v}{2}$. We can therefore see that there exists an (m_1, \dots, m_τ, N) -decomposition of λK_v (or $\lambda K_v - I$) since the hypotheses of Theorem 5 are satisfied by (i)–(iii) and the fact that $\sum N = \varepsilon$.

Case 2. Suppose v is even and $\varepsilon \in \{1, 2\}$. Let $M = m_1, \dots, m_\tau$ and let m be the least odd entry in M if M contains an odd entry, otherwise let m be the least entry in M such that $m \geq 4$ (such an entry exists by (iii)). Note that $v \geq 4$ and if $\varepsilon = 1$ then it follows from (ii) that M contains an odd entry and m is odd.

Case 2a. Suppose $m + \varepsilon \leq v$. By Theorem 5 there exists an $(M \setminus (m), m + \varepsilon)$ -decomposition \mathcal{D} of $\lambda K_v - I$, where I is a 1-factor of λK_v . Let \mathcal{P} be the $(M \setminus (m))$ -packing of λK_v that is obtained by removing an $(m + \varepsilon)$ -cycle from \mathcal{D} . Let L be the leave of \mathcal{P} and note that L consists of an $(m + \varepsilon)$ -cycle and the 1-factor I .

If L contains an $(m + \varepsilon, 1)$ -lasso then we apply Lemma 3 to \mathcal{P} with $s = m$ to complete the proof. The hypotheses of Lemma 3 are satisfied because $\varepsilon + 1 \geq 2$, and if m is even then M contains no odd entries so $\varepsilon = 2$ by (ii).

So suppose L does not contain an $(m + \varepsilon, 1)$ -lasso. Then $m + \varepsilon$ is even and L contains a component H such that H is the union of an $(m + \varepsilon)$ -cycle and a 1-factor on $V(H)$. We apply Lemma 4 to \mathcal{P} with $s = m + \varepsilon - 1$ to obtain an $(M \setminus (m))$ -packing \mathcal{P}' of λK_v whose leave L' contains a component H' on $m + \varepsilon$ vertices that has $\frac{3}{2}(m + \varepsilon)$ edges and contains an $(m + \varepsilon - 1, 1)$ -lasso. If $\varepsilon = 1$ then we can add the m -cycle of this lasso to \mathcal{P}' to complete the proof. Otherwise $\varepsilon = 2$ and H' contains an $(m + 1)$ -cycle with a chord because $m \geq 3$ and any vertex in this cycle has degree at least 3 (note that $\deg_H(x) = 3$ for each $x \in V(H)$). Then we can apply Lemma 4 with $s = m$ to \mathcal{P}' to obtain an $(M \setminus (m))$ -packing \mathcal{P}'' of λK_v whose leave contains an $(m, 1)$ -lasso. We add the m -cycle of this lasso to \mathcal{P}'' to complete the proof.

Case 2b. Suppose $m + \varepsilon > v$. Then $m \geq v - 1$ and $\varepsilon = 2$ (note that ε is even if $m = v$).

If $m = v$ then $m_i \in \{2, v\}$ for all $i \in \{1, \dots, \tau\}$, so $\lambda\binom{v}{2} - \frac{v}{2} \equiv 2 + \sum_{m_i=2} m_i \pmod{v}$ by (ii) and hence $2 + \sum_{m_i=2} m_i \leq (\lambda - 1)\binom{v}{2}$ by (iii). Then by Theorem 5 there exists an $(M, 2)$ -decomposition \mathcal{D} of $\lambda K_v - I$. We remove a 2-cycle from \mathcal{D} to complete the proof.

So suppose that $m = v - 1$. Since ε is even, M contains an even number of odd entries, so at least two entries of M are equal to $v - 1$. Let \mathcal{D}_0 be an $(M \setminus ((v - 1)^2), v^2)$ -decomposition of $\lambda K_v - I$ which exists by Theorem 5. Let \mathcal{P}_0 be the $(M \setminus ((v - 1)^2), v)$ -packing of λK_v formed by removing a v -cycle from \mathcal{D}_0 . The leave L_0 of \mathcal{P}_0 is the union of a v -cycle and the 1-factor I . Let \mathcal{P}_1 be the packing obtained by applying Lemma 4 to \mathcal{P}_0 with $s = v - 1$. Then the leave of \mathcal{P}_1 contains a $(v - 1, 1)$ -lasso. We add the $(v - 1)$ -cycle of this lasso to \mathcal{P}_1 and remove a v -cycle to obtain an $(M \setminus (v - 1))$ -packing \mathcal{P}_2 of λK_v . The leave of \mathcal{P}_2 has size $3\frac{v}{2} + 1$.

By applying Lemma 4 to \mathcal{P}_2 with $s = v - 1$ we obtain an $(M \setminus (v - 1))$ -packing \mathcal{P}_3 of λK_v whose leave contains a $(v - 1, 1)$ -lasso. We add the $(v - 1)$ -cycle of this lasso to \mathcal{P}_3 to complete the proof. \square

Lemma 8. Let m_1, \dots, m_τ be a nondecreasing list of integers and let λ and v be positive integers with λ even. Then there exists an (m_1, \dots, m_τ) -packing of λK_v if and only if

$$(i) \quad 2 \leq m_1, \dots, m_\tau \leq v;$$

$$(ii) \quad m_1 + \dots + m_\tau = \lambda\binom{v}{2} - \delta, \text{ where } \delta \text{ is a nonnegative integer such that } \delta \neq 1; \text{ and}$$

$$(iii) \quad m_\tau \leq \begin{cases} \frac{\lambda}{2}\binom{v}{2} - \tau + 2 & \text{if } \delta = 0, \\ \frac{\lambda}{2}\binom{v}{2} - \tau + 1 & \text{if } 2 \leq \delta < m_\tau. \end{cases}$$

Proof. If there exists an (m_1, \dots, m_τ) -packing \mathcal{P} of λK_v with leave L , then conditions (i)–(iii) hold by Lemma 6. So it remains to show that if λ , v and m_1, \dots, m_τ satisfy (i)–(iii), then there exists an (m_1, \dots, m_τ) -packing of λK_v . If $\delta = 0$ then the result follows immediately from Theorem 5, so suppose $\delta \geq 2$.

Let

$$N = \begin{cases} (\delta) & \text{if } 2 \leq \delta < m_\tau, \\ (2^{(\delta-m_\tau)/2}, m_\tau) & \text{if } \delta \geq m_\tau \text{ and } \delta \equiv m_\tau \pmod{2}, \\ (2^{(\delta-m_\tau+1)/2}, m_\tau - 1) & \text{if } \delta \geq m_\tau \text{ and } \delta \not\equiv m_\tau \pmod{2}. \end{cases}$$

Note that in each case $\sum N = \delta$. We show that there exists an (m_1, \dots, m_τ, N) -decomposition \mathcal{D} of λK_v because the hypotheses of Theorem 5 are satisfied by (i)–(iii) and the definition of N . The required packing is then obtained by removing cycles of lengths N from \mathcal{D} .

Let s be the number of entries in N . Let $M = m_1, \dots, m_\tau$. First observe that $\sum M + \sum N = \lambda \binom{v}{2}$ by (ii) and since $\sum N = \delta$. By (i) and the definition of N it also holds that $2 \leq n \leq m_\tau \leq v$ for all $n \in N$. If $2 \leq \delta < m_\tau$, then $m_\tau \leq \frac{\lambda}{2} \binom{v}{2} - \tau - s + 2$ by (iii) and since $s = 1$. If $\delta \geq m_\tau$, then because $\sum M \geq m_\tau + 2(\tau - 1)$ and $\sum N \geq m_\tau - 1 + 2(s - 1)$, it follows that

$$\begin{aligned} \frac{\lambda}{2} \binom{v}{2} - \tau - s + 2 &= \frac{1}{2}(\sum M + \sum N) - \tau - s + 2 \\ &\geq \frac{1}{2}(m_\tau + 2(\tau - 1) + m_\tau - 1 + 2(s - 1)) - \tau - s + 2 \\ &= m_\tau - \frac{1}{2}. \end{aligned}$$

Therefore $\max(N, M) = m_\tau \leq \frac{\lambda}{2} \binom{v}{2} - \tau - s + 2$ because $\frac{\lambda}{2} \binom{v}{2} - \tau - s + 2$ is an integer. So by Theorem 5 we can see that there exists an (M, N) -decomposition of λK_v which completes the proof. \square

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